Markets for Risk Management

Applications of Option Pricing Theory to Insurance

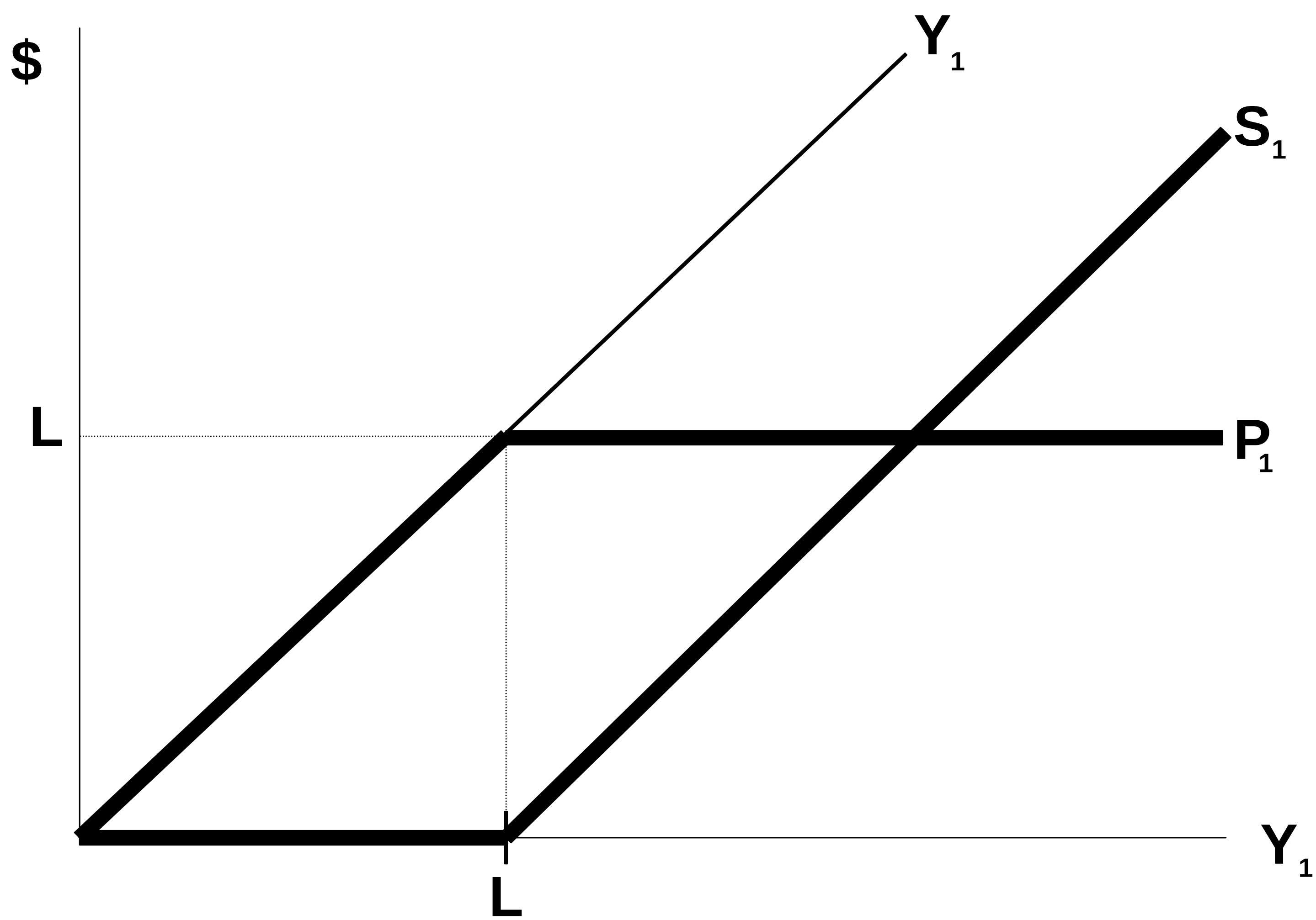
This lecture note is based primarily upon

Garven, James R., 2013, "Derivation and Comparative Statics of the Black-Scholes Call and Put Option Pricing Equations.

Economics of Limited Liability

- Assume a single period – the insurer is formed at t=0, and cash flows are realized one period later (at t=1).
- Y_0, P_0, and S_0 represent t=0 market values of assets, policyholder claims, and surplus, where $Y_0 = S_0 + P_0$.
- Y_1, P_1, and S_1 represent t=1 market values, where $Y_1 = P_1 + S_1 = (S_0 + P_0)(1 + r_i)$, $S_1 = Y_1 – P_1$, and $P_1 = L – \text{Max}[L - Y_1, 0]$.

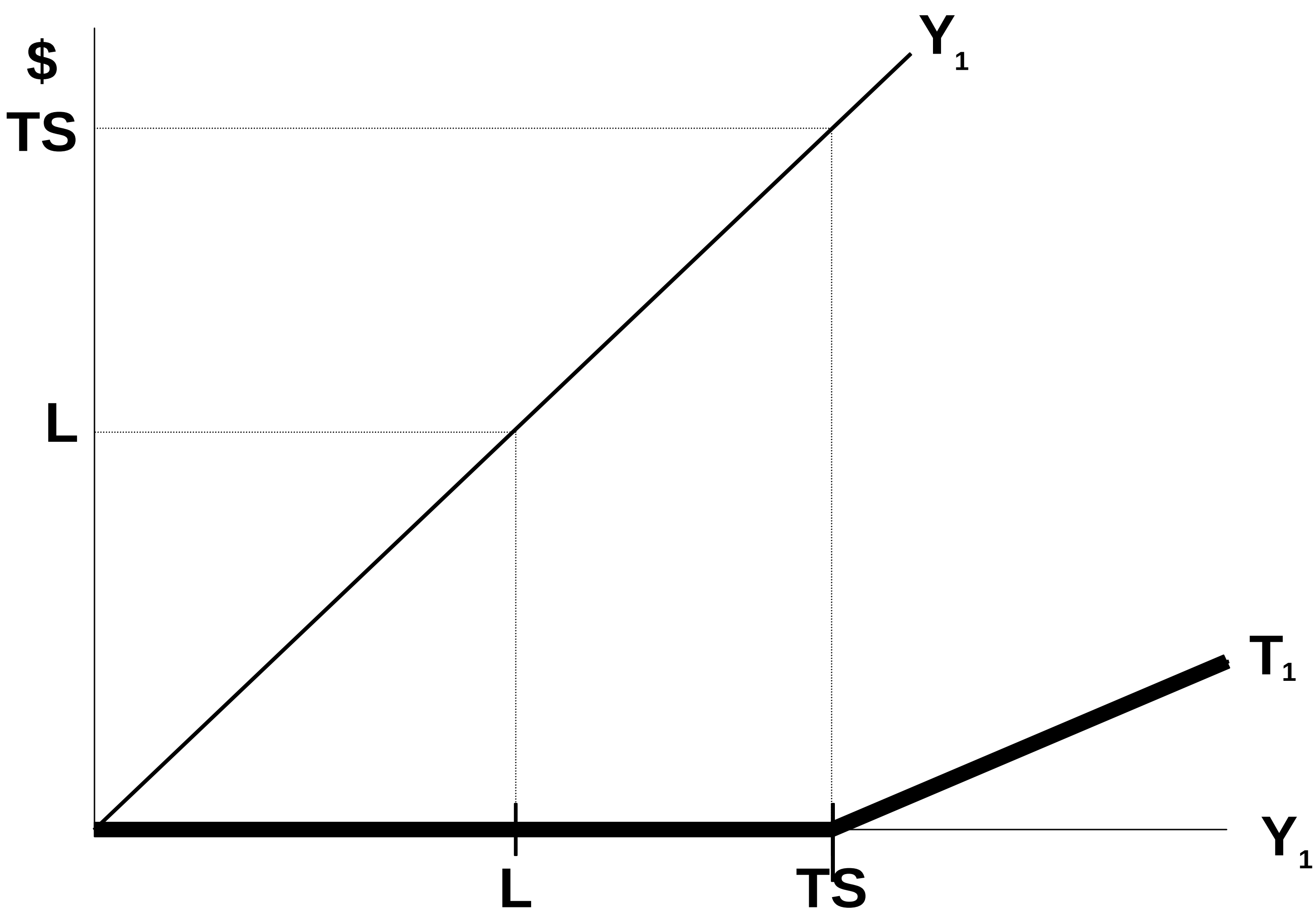
Economics of Limited Liability



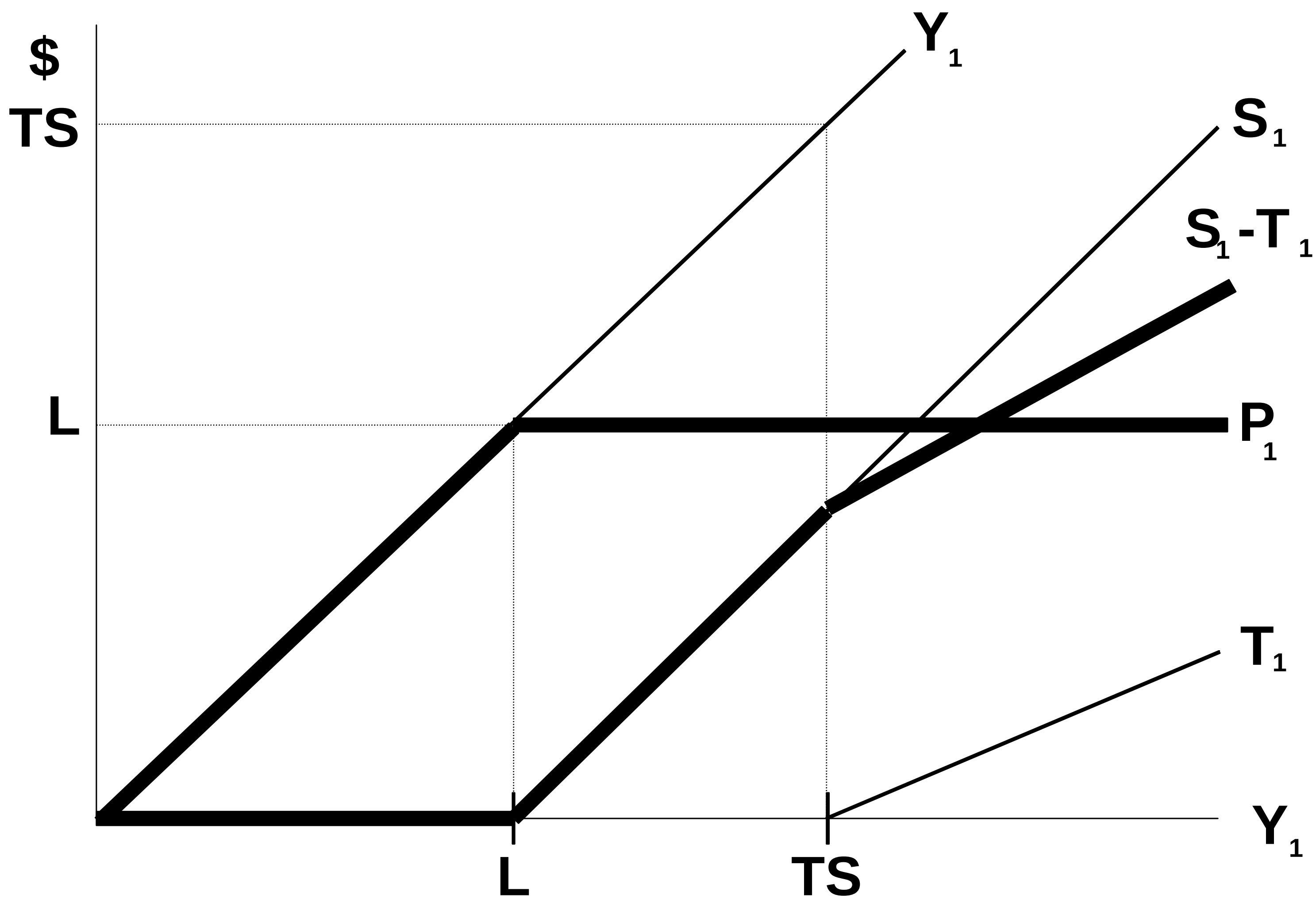
Asymmetric Taxes

- Insurers pay taxes (at rate τ) on underwriting profits and the taxable portion (θ) of investment income; i.e., $T_1 = \tau[\theta(Y_1 - Y_0) + (P_0 - L)] = \tau[Y_1 - TS]$, where $TS = L + S_0 + (1-\theta)r_i(S_0 + P_0)$.
- Furthermore, the government claims limited liability; therefore, $T_1 = \tau \text{Max}[Y_1 - TS, 0]$.

Asymmetric Taxes



Limited Liability and Asymmetric Taxes



Motivation

- Here, we provide an alternative derivation of the Black-Scholes-Merton call and put option pricing formulas using an *integration* rather than *differential equations* approach.
- The integration approach clarifies the economics and mathematics of option pricing theory and conveys a deeper and better *intuitive* understanding of option pricing theory and its applications using basic calculus and statistics.
- Comparative statics are also derived.

Risk Neutral Valuation Relationship

- Definition: A risk-neutral valuation relationship (*RNVR*) exists if the relationship between the price of an derivative security (e.g., an option) and the price of its underlying asset does not depend upon investor risk preferences.
- Black-Scholes-Merton's (BSM's) *key* insight was that by dynamically hedging a long (short) call with a short (long) stock position, investors create riskless hedge portfolios which imply a specific type of *RNVR*.
  - Given this *RNVR*, for a given price of the underlying stock, there exists a unique value for the option that is implied by the *RNVR*.
- An alternative path to an *RNVR* involves imposing restrictions on investor preferences and asset price distributions; here, we focus our attention on the dynamic hedging path chosen by BSM.

Geometric Brownian Motion

- Black and Scholes assume that stock prices change continuously according to the Geometric Brownian Motion equation; i.e.,

$$dS = \mu S dt + \sigma S dz. \quad (1)$$

where $dz = \varepsilon\sqrt{dt}$, ϵ is a standard normal random variable, dS is the stock price change per dt time unit, S is the current stock price, μ is the expected return, and σ represents volatility.

Ito's Lemma

- At any given point in time, the value of the call option (C) depends upon the value of the underlying asset; i.e., $C = C(S,t)$.
- Ito's Lemma justifies the use of a Taylor-series-like expansion for the differential dC:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2. \quad (2)$$

- Since $dS^2 = S^2\sigma^2 dt$, substituting for dS^2 in equation yields equation:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}dt. \quad (3)$$

Ito's Lemma

- $V = C(S,\ t)$ - $\Delta_t S$, implying that

$$dV = dC - \Delta_t dS = \underbrace{\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt}_{\text{deterministic}} + \underbrace{\left(\frac{\partial C}{\partial S} - \Delta_t\right) dS.}_{\text{stochastic}} \quad (4)$$

- If $\partial C/\partial S = \Delta_t$, then

$$dV = dC - \Delta_t dS = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt. \quad (5)$$

The Black-Scholes-Merton RNVR

- In order to prevent arbitrage, the hedge portfolio must earn the riskless rate of interest r; i.e.,

$$dV = rV\,dt. \tag{6}$$

- $\Delta_t = \dfrac{\partial C}{\partial S}$ implies that $V = C - \dfrac{\partial C}{\partial S} S$. Substituting $C - \dfrac{\partial C}{\partial S} S$ in place of V on the right-hand side of equation (6) and equating this with the right-hand side of equation (5), we obtain:

$$r\left(C - S\frac{\partial C}{\partial S}\right) dt = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt. \tag{7}$$

The Black-Scholes-Merton RNVR

- Dividing both sides of equation (7) by dt and rearranging results in the Black-Scholes-Merton (non-stochastic) partial differential equation:

$$rC = \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S}. \tag{8}$$

- Equation (8) shows that the valuation relationship between a call option and its underlying asset is *deterministic*.
  - Since risk preferences play no role in equation (8), this implies that the price of a call option can be calculated *as if* investors are *risk neutral*.
- Today's call option price (C) must satisfy equation (8), subject to the constraint (or "boundary condition") that $C_t = Max[S_t - X, 0]$.

Solving the Black-Scholes-Merton RNVR for the option price

- Black-Scholes transform equation (8) into a heat transfer equation and employ a solution procedure from a textbook on applications of Fourier series to boundary value problems in engineering and physics, resulting in the following equation for the value of a European call option on a (non-dividend paying) stock:

$$C = SN(d_1) - Xe^{-rt}N(d_2), \tag{9}$$

where

$$d_1 = \frac{\ln(S/X) + (r + .5\sigma^2)t}{\sigma\sqrt{t}};$$

$$d_2 = d_1 - \sigma\sqrt{t};$$

σ^2= variance of underlying asset's rate of return; and

$N(z)$ = standard normal distribution function evaluated at z.

Option Pricing via Integration

- The value today (C) of a European call option that pays $C_t = Max[S_t - X, 0]$ at date t is given by the following equation:

$$C = V(C_t) = V(Max[S_t - X, 0]). \tag{10}$$

The valuation operator $V(\cdot)$ determines the call option price by discounting the *risk neutral* expected value of the option's payoff at expiration ($\hat{E}(C_t)$) at the riskless rate of interest:

$$C = e^{-rt}\hat{E}(C_t) = e^{-rt}\int_X^{\infty}(S_t - X)\hat{h}(S_t)dS_t, \tag{11}$$

where $\hat{h}(S_t)$ represents the risk neutral lognormal density function of S_t.

- We'll start by calculating the expected value of C_t ($E(C_t)$), rather than its *risk neutral* expected value ($\hat{E}\left(C_t\right)$):

$$E(C_t) = E[Max(S_t - X, 0)] = \int_X^\infty (S_t - X)h(S_t)dS_t, \quad (12)$$

where $h(S_t)$ represents $S_t{}'\mathrm{s}$ lognormal density function.

- Statistical Note: The main difference between the $\hat{h}(S_t)$ and $h(S_t)$ density functions is that the location parameter for $h(S_t)$ is μt, whereas it is $(r - .5\sigma^2)t$ for $\hat{h}(S_t)$ This is conceptually similar to the relationship between the *actual* probability of an "up" move in the binomial model compared with the corresponding *risk neutral* probability of an "up" move.

Option Pricing via Integration

- Next, we evaluate the integral given by equation (12) by rewriting it as the difference between two integrals:

$$\begin{aligned} E(C_t) &= \int_X^\infty S_t h(S_t) dS_t - X \int_X^\infty h(S_t) dS_t \\ &= E_X(S_t) - X e^{-rt} \left[1 - H(X)\right]. \end{aligned} \quad (13)$$

- Next, we define the t-period lognormally distributed price ratio as $R_t = S_t/S$. Thus, $S_t = S\,(R_t)$, and we rewrite equation (13) as

$$\begin{aligned} E(C_t) &= S \int_{X/S}^\infty R_t g(R_t) dR_t - X \int_{X/S}^\infty g(R_t) dR_t \\ &= S E_{X/S}(R_t) - X[1 - G(X/S)], \end{aligned} \quad (14)$$

Option Pricing via Integration

- Next, consider the partial expected value of the terminal stock price, $SE_{X/S}(R_t)$. Note that:
  - $R_t = e^{kt}$, where *k* is the rate of return on the underlying asset per unit of time.
  - $\ln(R_t) = kt$ is normally distributed with density $f(kt)$, mean $\mu_k t$ and variance $\sigma_k^2 t$.
  - Since $g(R_t) = (1/R_t)f(kt)$ and $dR_t = e^{kt}tdk$, it follows that $R_t g(R_t) dR_t = e^{kt} f(kt) t dk$; thus,

$$\begin{aligned} SE_{X/S}(R_t) &= S\int_{\ln(X/S)}^{\infty} e^{kt} f(kt) t dk \\ &= S\frac{1}{\sqrt{2\pi\sigma^2 t}}\int_{\ln(X/S)}^{\infty} e^{kt} e^{-\{.5[(kt-\mu t)^2/\sigma^2 t]\}} t dk. \quad (15) \end{aligned}$$

Option Pricing via Integration

- Note that

$$
\begin{aligned}
e^{kt}e^{-\{.5[(kt-\mu t)^2/\sigma^2 t]\}} &= e^{-\{.5t[(k^2-2\mu k+\mu^2-2\sigma^2 k)/\sigma^2]\}} \\
&= e^{-\{.5t[(k^2-2\mu k+\mu^2-2\sigma^2 k+\sigma^4-\sigma^4)/\sigma^2]\}} \\
&= e^{-\{.5t[((k-\mu-\sigma^2)^2-\sigma^4-2\mu\sigma^2)/\sigma^2]\}} \\
&= e^{(\mu+.5\sigma^2)t}e^{-\{.5[(kt-(\mu+\sigma^2)t)^2/\sigma^2 t]\}}. \quad (16)
\end{aligned}
$$

- In (16), $e^{(\mu+.5\sigma^2)t} = E(R_t)$!

Option Pricing Formula Derivation

- Therefore,

$$SE_{X/S}(R_t) = SE(R_t) \times$$

$$\frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\ln(X/S)}^{\infty} e^{-\{.5[(kt-(\mu+\sigma^2)t)^2/\sigma^2 t]\}} t dk$$

$$= E(S_t) \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\ln(X/S)}^{\infty} e^{-\{.5[(kt-(\mu+\sigma^2)t)^2/\sigma^2 t]\}} t dk. \quad (17)$$

- Next, let $y = [kt - (\mu + \sigma^2)t] / \sigma\sqrt{t} \Rightarrow kt =$ $(\mu + \sigma^2)t + \sigma\sqrt{t}\, y$ and $tdk = \sigma\sqrt{t}\, dy$.

Option Pricing Formula Derivation

- Thus, (18) follows:

$$SE_{X/S}(R_t) = E(S_t) \int_{\frac{\ln(X/S)-(\mu+\sigma^2)t}{\sigma\sqrt{t}}}^{\infty} [e^{-.5y^2} / \sqrt{2\pi}]dy$$

$$= E(S_t)\int_{-\delta_1}^{\infty} n(y)dy = E(S_t)\int_{-\infty}^{\delta_1} n(y)dy$$

$$= E(S_t)N(\delta_1),$$

where $N(\delta_1)$ is the standard normal distribution function evaluated at $y = \delta_1$.

Option Pricing Formula Derivation

- Next, consider $X\int_{X/S}^{\infty} g(R_t)dR_t$ (see (14)). Since $g(R_t)dR_t = f(kt)tdk$, (19) obtains:

$$X\int_{X/S}^{\infty} g(R_t)dR_t = X\int_{\ln(X/S)}^{\infty} f(kt)tdk$$

$$= X\frac{1}{\sqrt{2\pi\sigma^2 t}}\int_{\ln(X/S)}^{\infty} e^{-\{.5[(kt-\mu t)^2/\sigma^2 t]\}}tdk. \quad (19)$$

Option Pricing Formula Derivation

- Let $z = [kt - \mu t]/\sigma\sqrt{t} \Rightarrow kt = \mu t + \sigma\sqrt{t}z$ and $tdk = \sigma\sqrt{t}dz \Rightarrow$ limit of integration is $[\ln(X/S) - \mu t]/\sigma\sqrt{t}$ = -(δ_1 - $\sigma\sqrt{t}$) $= -\delta_2$.
- Thus, (20) obtains:

$$X\int_{X/S}^{\infty} g(R_t)dR_t$$

$$= X\int_{-\delta_2}^{\infty} [e^{-.5z^2} / \sqrt{2\pi}]dz \qquad (20)$$

$$= X\int_{-\infty}^{\delta_2} n(z)dz = XN(\delta_2).$$

Option Pricing Formula Derivation

- Substituting (18) and (20) into (14) yields (21):

$$E(C_t) = E(S_t)N(\delta_1) - XN(\delta_1 - \sigma\sqrt{t}). \quad (21)$$

- Since $C = e^{-rt}\hat{E}\left(C_t\right) = e^{-rt}\hat{E}[Max(S_t - X, 0)]$, we need to determine risk neutral values for $E(S_t)$ and δ_1.

- Since $(\mu + .5\sigma^2)t = rt$ in a risk neutral economy,

$$\hat{E}(S_t) = Se^{rt};\ \hat{\delta}_1 = d_1 = \frac{\ln(S/X) + (r + .5\sigma^2)t}{\sigma\sqrt{t}}.$$

Option Pricing Formula Derivation

- Substituting (21) into (11) and simplifying yields the Black-Scholes call option pricing formula:

$$
\begin{aligned}
C &= e^{-rt}\hat{E}\left(C_t\right) \\
&= e^{-rt}\left[Se^{rt}N(d_1) - XN(d_1 - \sigma\sqrt{t})\right] \qquad (22) \\
&= SN(d_1) - Xe^{-rt}N(d_2).
\end{aligned}
$$

Option Pricing Formula Derivation

- The put option pricing formula follows directly from the put-call parity theorem:

$$\begin{aligned} P &= C + Xe^{-rt} - S \\ &= SN(d_1) - Xe^{-rt}N(d_2) + Xe^{-rt} - S \\ &= Xe^{-rt}\left[1 - N(d_2)\right] - S\left[1 - N(d_1)\right] \\ &= Xe^{-rt}N(-d_2) - SN(-d_1). \end{aligned} \tag{23}$$

Comparative Statics

- What is the call option hedge ratio ($\partial C/\partial S$; aka "delta")?

$$\partial C/\partial S = N(d_1)+S(\partial N(d_1)/\partial d_1)(\partial d_1/\partial S) - Xe^{-rt}(\partial N(d_2)/\partial d_2)(\partial d_2/\partial S)$$

$$= N(d_1)+Sn(d_1)(\partial d_1/\partial S) - Xe^{-rt}n(d_2)(\partial d_2/\partial S) \quad (24)$$

Substituting $d_2 = d_1 - \sigma\sqrt{t}$, $\partial d_2/\partial S = \partial d_1/\partial S$ and $n(d_2) = n(d_1 - \sigma\sqrt{t})$, (25) obtains:

Comparative Statics

$$\partial C/\partial S = N(d_1)+(\partial d_1/\partial S)[Sn(d_1)\text{-}Xe^{-rt}n(d_1 \text{-}\sigma\sqrt{t})]$$

$$= N(d_1) + (\partial d_1/\partial S)\frac{1}{\sqrt{2\pi}}[Se^{-.5d_1^2}\text{-}X\,e^{-rt}e^{-.5(d_1-\sigma\sqrt{t})^2}] \quad (25)$$

Since $d_1 = [\ln(S/X) + (r+.5\sigma^2)t]/\sigma\sqrt{t}$, $S = Xe^{d_1\sigma\sqrt{t}-(r+.5\sigma^2)t}$. Substituting for S in (25)'s bracketed term yields:

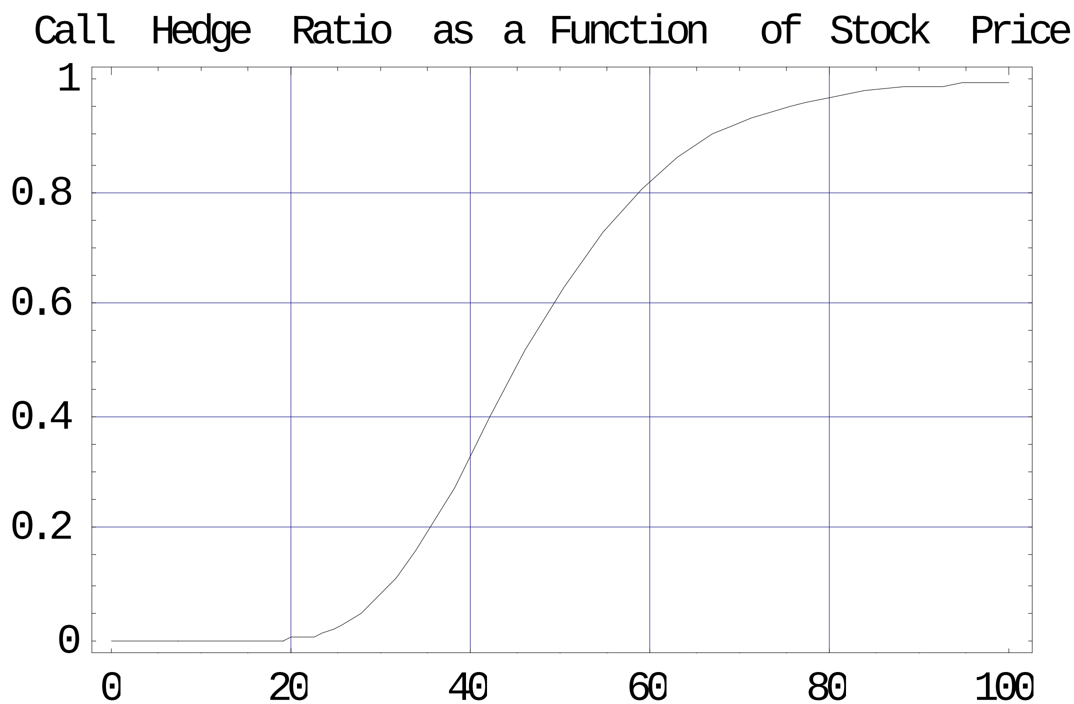
$$\partial C/\partial S = N(d_1) +$$

$$\frac{\partial d_1/\partial S}{\sqrt{2\pi}}[Xe^{-.5d_1^2}e^{d_1\sigma\sqrt{t}-(r+.5\sigma^2)t}\text{-}X\,e^{-rt}e^{-.5(d_1-\sigma t^{.5})^2}])]$$

Comparative Statics

$$= N(d_1) + \frac{\partial d_1 / \partial S}{\sqrt{2\pi}} [X(e^{-(r+.5\sigma^2)t + d_1\sigma\sqrt{t} - .5d_1^2}$$

$$- e^{-(r+.5\sigma^2)t + d_1\sigma\sqrt{t} - .5d_1^2})] = N(d_1) > 0 \ (26)$$



Comparative Statics

- What is the put option hedge ratio ($\partial P/\partial S$)?

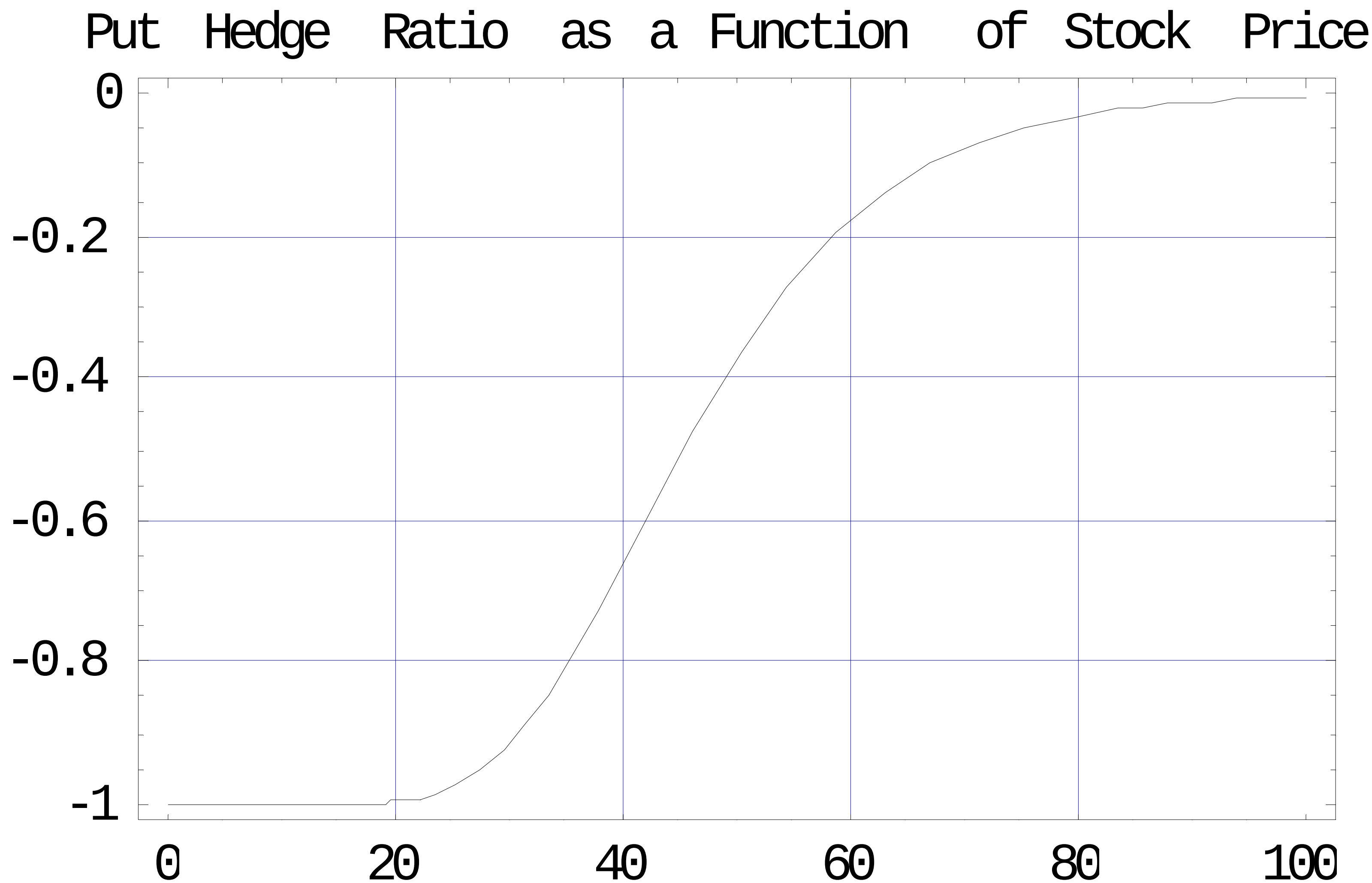
$$\partial P/\partial S = -N(-d_1)+Xe^{-rt}(\partial N(-d_2)/\partial d_2)(\partial d_2/\partial S) - S(\partial N(-d_1)/\partial d_1)(\partial d_1/\partial S)$$

$$= -N(-d_1)-Xe^{-rt}n(-d_2)(\partial d_1/\partial S)+Sn(-d_1)(\partial d_1/\partial S)$$

$$= -N(-d_1)+ (\partial d_1/\partial S)[Sn(-d_1) - Xe^{-rt}n(\sigma\sqrt{t}-d_1)] \quad (27)$$

Since $Sn(-d_1) - Xe^{-rt}n(\sigma\sqrt{t}-d_1) = Sn(d_1) - Xe^{-rt}n(d_1 - \sigma\sqrt{t}) = 0$, $\partial P/\partial S = -N(-d_1) < 0$.

Comparative Statics



Comparative Statics

- Note that the call option delta is $N(d_1)$, whereas the put option delta $-N(-d_1) = N(d_1)-1$!

Call Delta ($N(d_1)$)	**Put Delta ($-N(-d_1)$)**
1	0
.8	-.2
.6	-.4
.4	-.6
.2	-.8
0	-1

Comparative Statics

- How about $\partial C/\partial X$?

$$\partial C/\partial X = -e^{-rt}N(d_2) + S(\partial N(d_1)/\partial d_1)(\partial d_1/\partial X) - Xe^{-rt}(\partial N(d_2)/\partial d_2)(\partial d_2/\partial X)$$

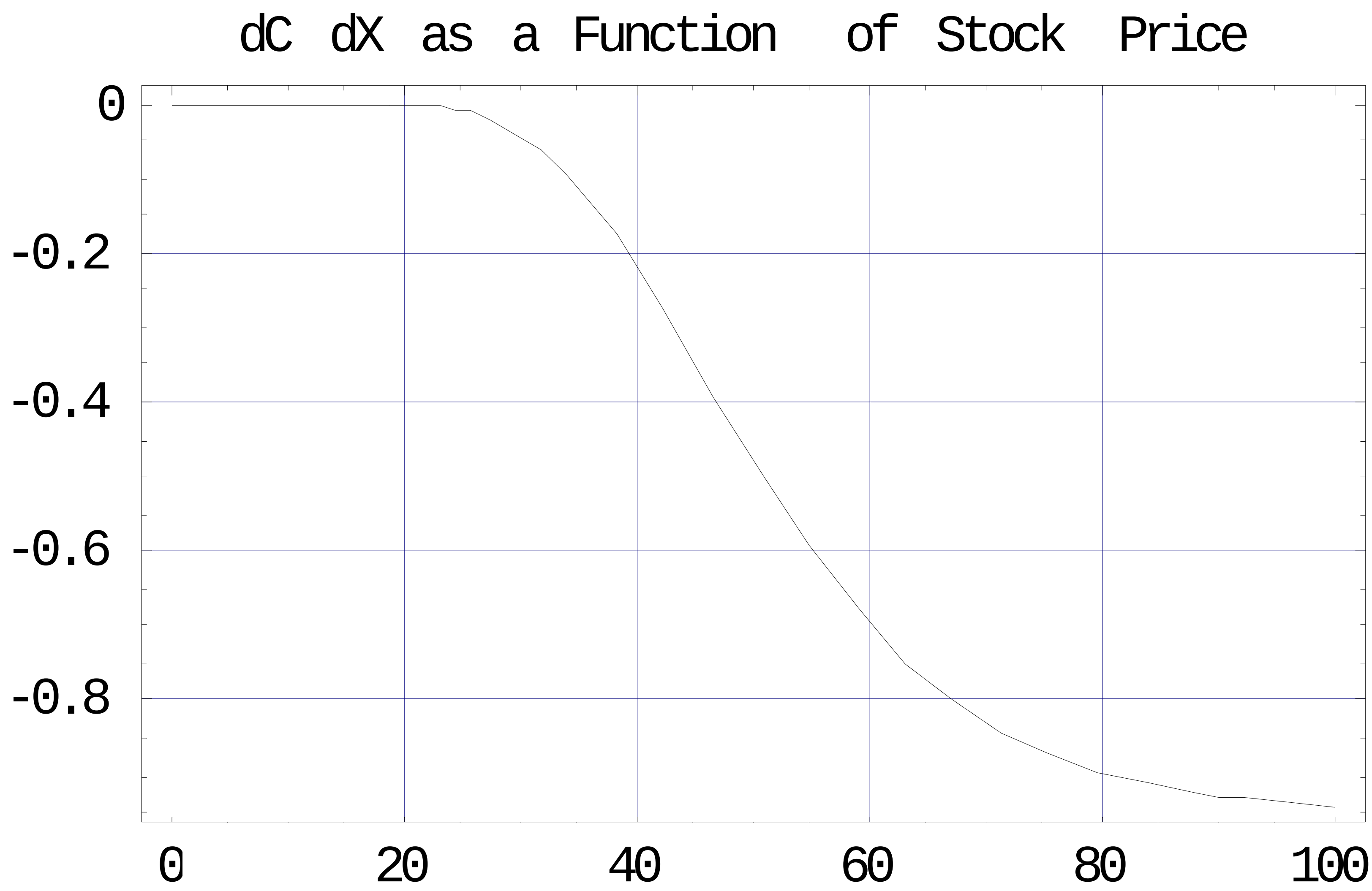
$$= -e^{-rt}N(d_2) + Sn(d_1)\frac{\partial d_1}{\partial X} - Xe^{-rt}n(d_2)\frac{\partial d_2}{\partial X} \qquad (28)$$

Substituting $d_2 = d_1 - \sigma\sqrt{t}$, $\partial d_2/\partial X = \partial d_1/\partial X$ and $n(d_2) = n(d_1 - \sigma\sqrt{t})$, (29)' obtains:

$$\partial C/\partial X = -e^{-rt}N(d_2) + \frac{\partial d_1}{\partial X}[Sn(d_1) - Xe^{-rt}n(d_1 - \sigma\sqrt{t})]$$

$$= -e^{-rt}N(d_2) < 0. \qquad (29)'$$

Comparative Statics



Comparative Statics

- How about $\partial P/\partial X$?

$$\partial P/\partial X = e^{-rt}N(\text{-}d_2) + Xe^{-rt}\left(\frac{\partial N(-d_2)}{\partial d_2}\frac{\partial d_2}{\partial X}\right)$$

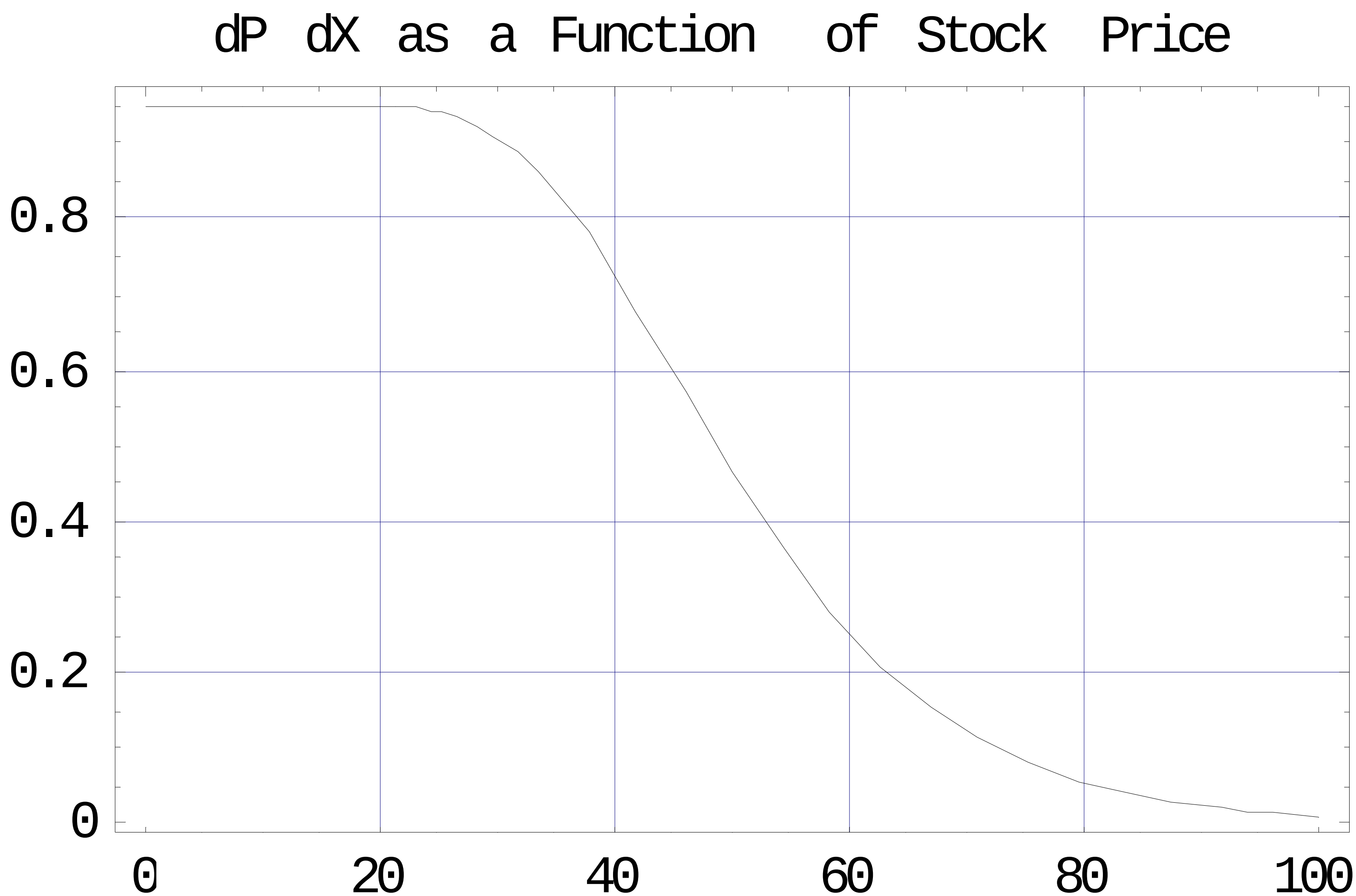
$$- S\left(\frac{\partial N(-d_1)}{\partial d_1}\frac{\partial d_1}{\partial X}\right)$$

$$= e^{-rt}N(\text{-}d_2) - Xe^{-rt}n(\text{-}d_2)\frac{\partial d_1}{\partial X} + Sn(\text{-}d_1)(\partial d_1/\partial X)$$

$$= e^{-rt}N(\text{-}d_2) + (\partial d_1/\partial X)[Sn(\text{-}d_1) - Xe^{-rt}n(\sigma\sqrt{t}\text{-}d_1)]$$

$$= e^{-rt}N(\text{-}d_2) > 0. \qquad (30)'$$

Comparative Statics



Comparative Statics

- How about $\partial C/\partial r$ (aka "rho")?

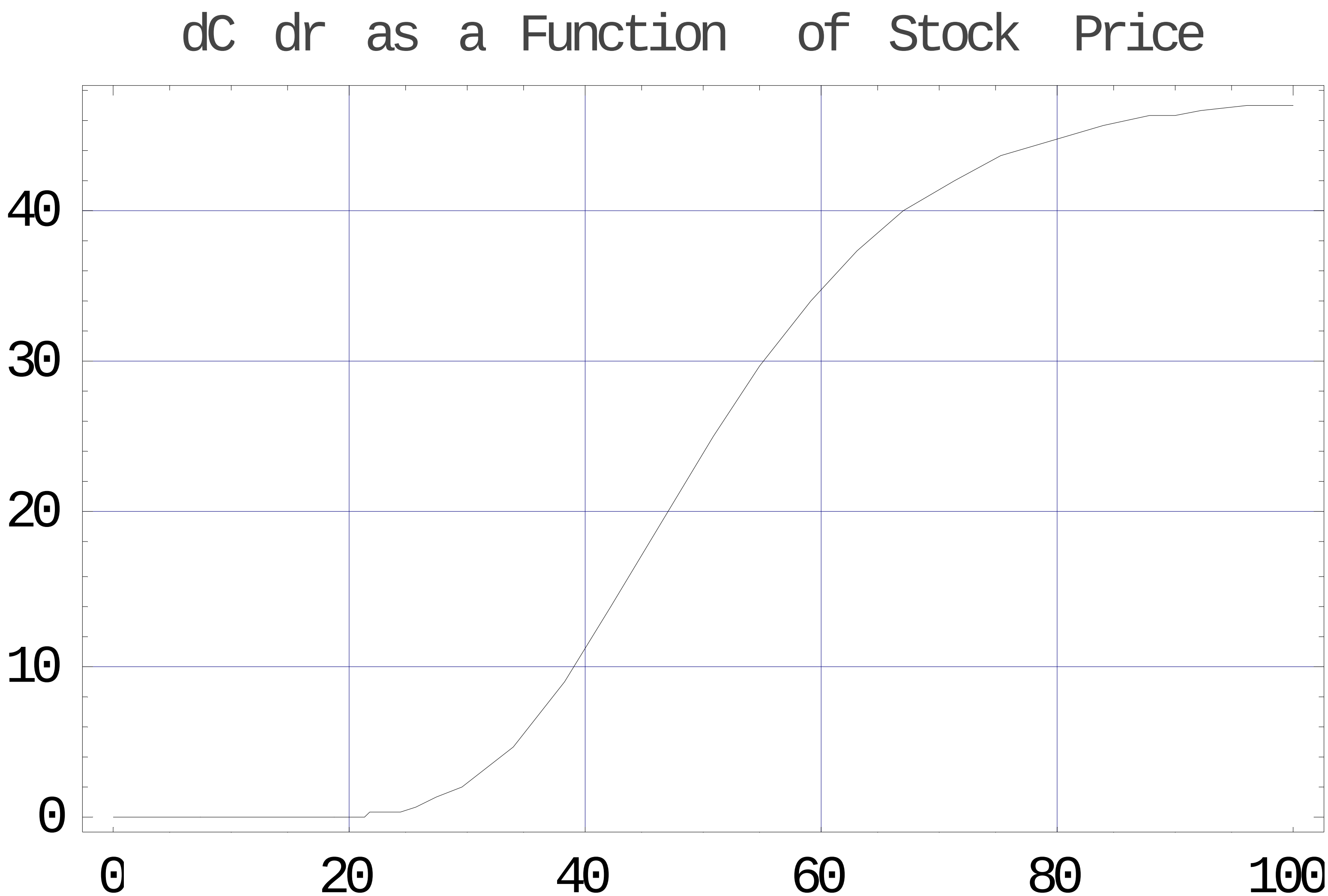
$$\partial C/\partial r = tXe^{-rt}N(d_2)+S(\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial r}) - Xe^{-rt}(\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial r})$$

$$= tXe^{-rt}N(d_2) + Sn(d_1)\frac{\partial d_1}{\partial r} - Xe^{-rt}n(d_2)\frac{\partial d_1}{\partial r}$$

$$= tXe^{-rt}N(d_2) + \frac{\partial d_1}{\partial r}[Sn(d_1) - Xe^{-rt}n(d_1 - \sigma\sqrt{t})]$$

$$= tXe^{-rt}N(d_2) > 0. \qquad (31)$$

Comparative Statics



Comparative Statics

- How about $\partial P/\partial r$?

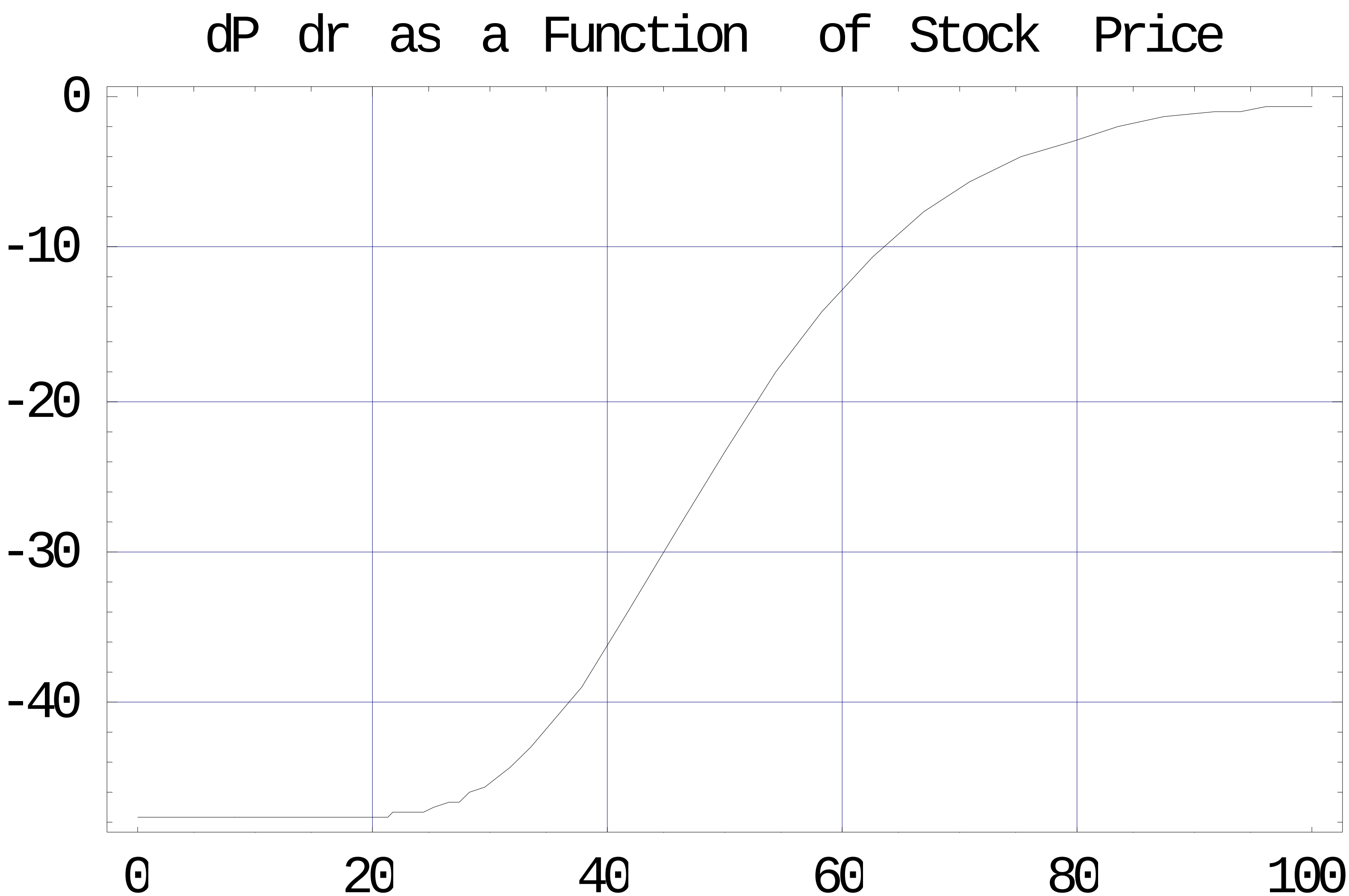
$$\frac{\partial P}{\partial r} = \text{-}tXe^{-rt}N(\text{-}d_2) + Xe^{-rt}\left(\frac{\partial N(-d_2)}{\partial d_2}\frac{\partial d_2}{\partial r}\right) \text{-} S\left(\frac{\partial N(-d_1)}{\partial d_1}\frac{\partial d_1}{\partial r}\right)$$

$$= \text{-}tXe^{-rt}N(\text{-}d_2) \text{ - } Xe^{-rt}n(\text{-}d_2)\frac{\partial d_1}{\partial r} + Sn(\text{-}d_1)\frac{\partial d_1}{\partial r}$$

$$= \text{-}tXe^{-rt}N(\text{-}d_2) + \frac{\partial d_1}{\partial r}[Sn(\text{-}d_1) \text{ - } Xe^{-rt}n(\sigma\sqrt{t}\text{-}d_1)]$$

$$= \text{-}tXe^{-rt}N(\text{-}d_2) < 0. \qquad (32)$$

Comparative Statics



Comparative Statics

- How about $\partial C/\partial t$ (aka "theta")?

$$\frac{\partial C}{\partial t} = rXe^{-rt}N(d_2) + S(\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial t}) - Xe^{-rt}(\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial t}) \quad (33)$$

Substituting $X = Se^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t}$ into (33),

$$\frac{\partial C}{\partial t} = rXe^{-rt}N(d_2) + S[n(d_1)\frac{\partial d_1}{\partial t} - e^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t-rt}n(d_2)\frac{\partial d_2}{\partial t}]$$

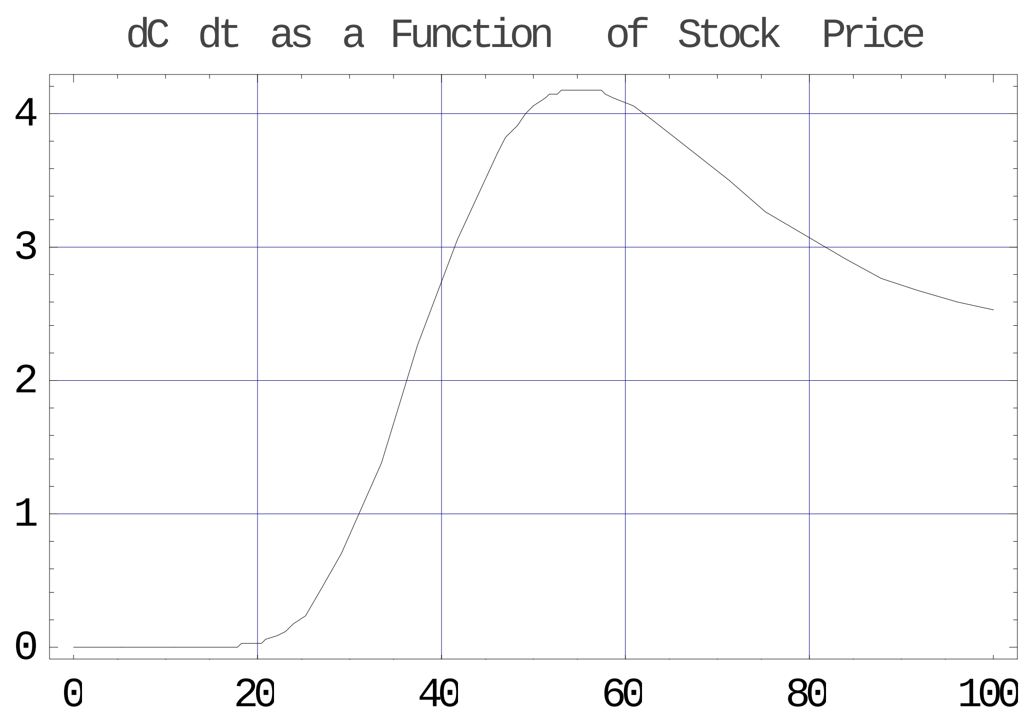
$$= rXe^{-rt}N(d_2) + \frac{S}{\sqrt{2\pi}}[\frac{\partial d_1}{\partial t}e^{-.5d_1^2} - \frac{\partial d_2}{\partial t}e^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t-rt-.5(d_1-\sigma\sqrt{t})^2}]$$

$$= rXe^{-rt}N(d_2) + \frac{S}{\sqrt{2\pi}}[\frac{\partial d_1}{\partial t}e^{-.5d_1^2} - \frac{\partial d_2}{\partial t}e^{-d_1\sigma\sqrt{t}+d_1\sigma\sqrt{t}+rt-rt+.5\sigma^2t-.5\sigma^2t-.5d_1^2}]$$

Comparative Statics

$$= rXe^{-rt}N(d_2) + Sn(d_1)[\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}] = rXe^{-rt}N(d_2) + Sn(d_1)\frac{.5\sigma}{\sqrt{t}}$$

Thus (as indicated in (34) above), $\partial C / \partial t > 0$.



Comparative Statics

- How about $\partial P/\partial t$? Consider equation (35):

$$\frac{\partial P}{\partial t} = -rXe^{-rt}N(-d_2)-Xe^{-rt}\left(\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial t}\right)+S\left(\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial t}\right) \quad (35)$$

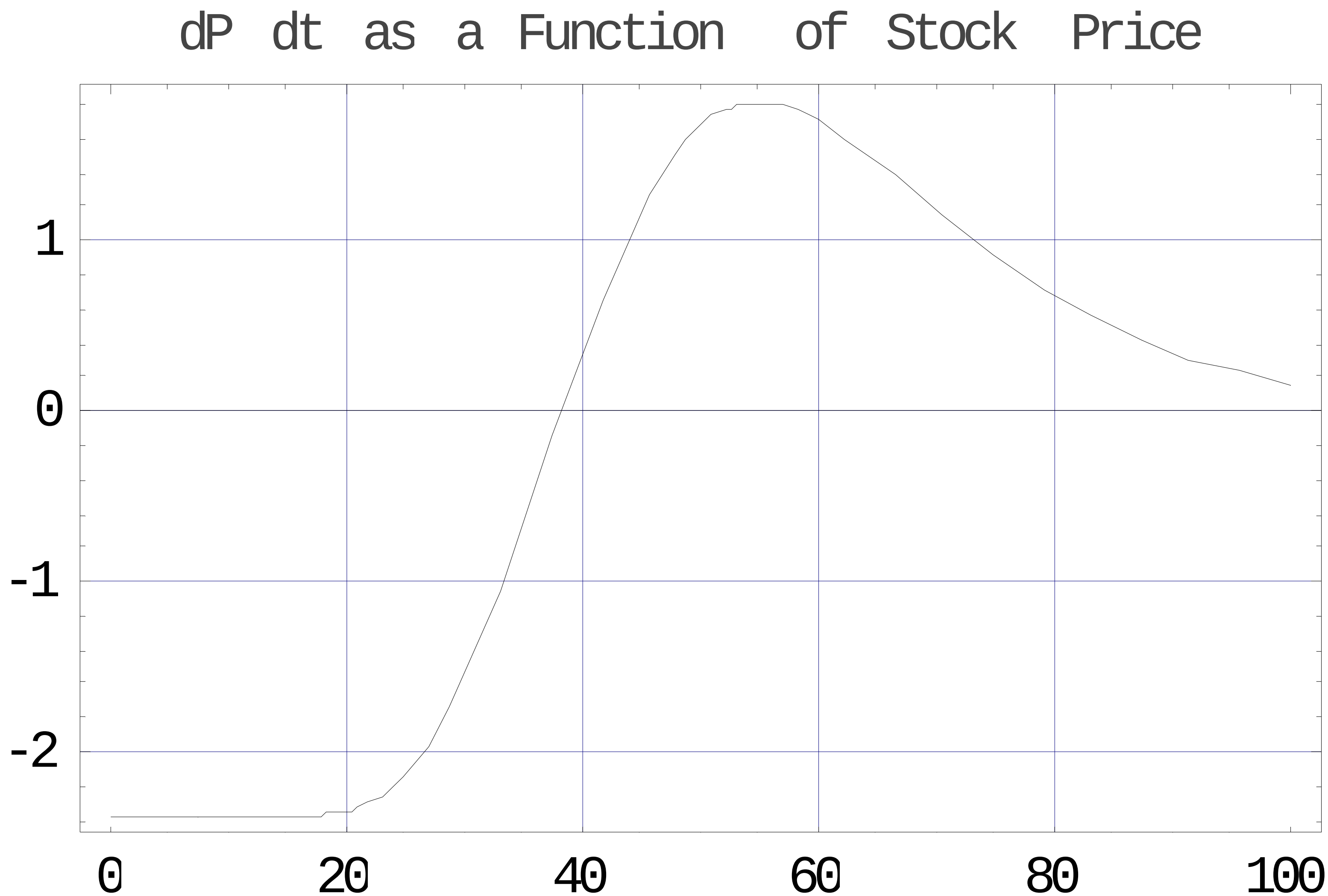
Substituting $X = Se^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t}$ into (35),

$$\frac{\partial P}{\partial t} = -rXe^{-rt}N(-d_2) - Se^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t-rt}n(d_2)\frac{\partial d_2}{\partial t} + Sn(d_1)\frac{\partial d_1}{\partial t}$$

$$= -rXe^{-rt}N(-d_2) - \frac{S}{\sqrt{2\pi}}e^{-d_1\sigma\sqrt{t}+.5\sigma^2 t-.5(d_1-\sigma\sqrt{t})^2}\frac{\partial d_2}{\partial t} + Sn(d_1)\frac{\partial d_1}{\partial t}$$

$$= -rXe^{-rt}N(-d_2) - Sn(d_1)\left[\frac{\partial d_2}{\partial t} - \frac{\partial d_1}{\partial t}\right] = -rXe^{-rt}N(-d_2)+Sn(d_1)\frac{.5\sigma}{\sqrt{t}}.$$

Comparative Statics



Comparative Statics

- How about $\partial C/\partial \sigma$ (aka "vega")?

$$\frac{\partial C}{\partial \sigma} = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \sigma} - Xe^{-rt}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma}. \qquad (36)$$

Substituting $\frac{\partial N(d_2)}{\partial d_2} = n(d_2)$, $\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{t}$, and $X =$ $Se^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t}$ into (36),

$$\frac{\partial C}{\partial \sigma} = S\left[n(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t-rt}\,n(d_2)\frac{\partial d_2}{\partial \sigma}\right]$$

Comparative Statics

$$= Sn(d_1)\frac{\partial d_1}{\partial \sigma} - Se^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t-rt}\frac{e^{-.5\left(d_1-\sigma\sqrt{t}\right)^2}}{\sqrt{2\pi}}\left(\frac{\partial d_1}{\partial \sigma} - \sqrt{t}\right) \qquad (37)$$

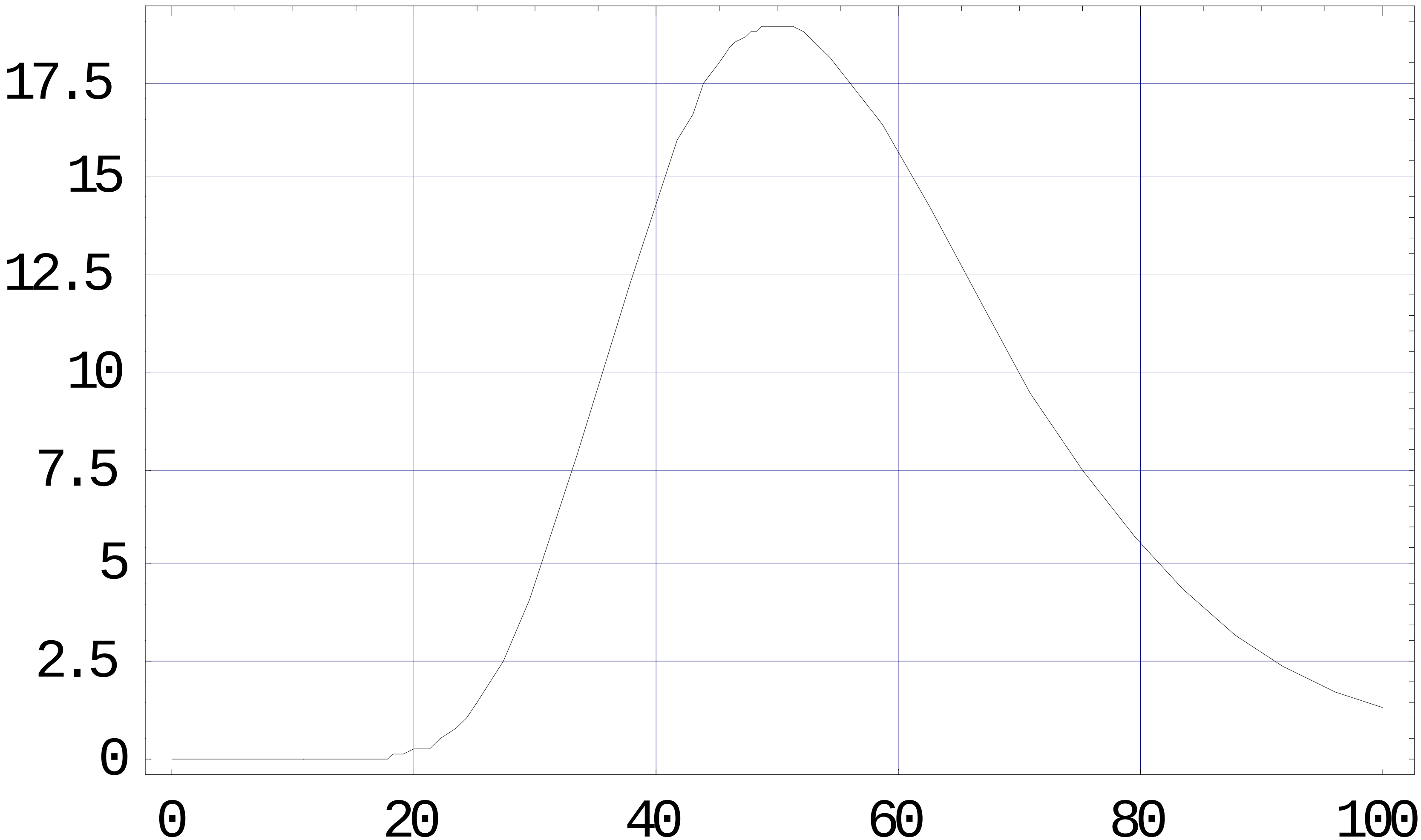
Since $e^{-d_1\sigma\sqrt{t}+rt+.5\sigma^2t-rt-.5\left(d_1-\sigma\sqrt{t}\right)^2} = e^{-.5d_1^2}$, equation (37) can be rewritten as

$$\frac{\partial C}{\partial \sigma} = Sn(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \left(\frac{\partial d_1}{\partial \sigma} - \sqrt{t}\right)\right] = Sn(d_1)\sqrt{t} \qquad (38)$$

Thus (as indicated in (38) above), $\partial C/\partial \sigma > 0$.

Comparative Statics

dC ds [redacted]ega [redacted]a Function of Stock Price



Comparative Statics

- How about $\partial P/\partial\sigma$?

$$\frac{\partial P}{\partial\sigma} = Xe^{-rt}\frac{\partial N(-d_2)}{\partial d_2}\frac{\partial d_2}{\partial\sigma} - S\frac{\partial N(-d_1)}{\partial d_1}\frac{\partial d_1}{\partial\sigma}$$

$$= -Xe^{-rt}n(\sigma\sqrt{t}-d_1)\frac{\partial d_2}{\partial\sigma} + Sn(-d_1)\frac{\partial d_1}{\partial\sigma}. \qquad (39)$$

Substituting $-d_2 = \sigma\sqrt{t} - d_1, \frac{\partial d_2}{\partial\sigma} = \frac{\partial d_1}{\partial\sigma} - \sqrt{t}$, and $X = Se^{-d_1\sigma\sqrt{t}+(r+.5\sigma^2)t}$ into (39) yields:

Comparative Statics

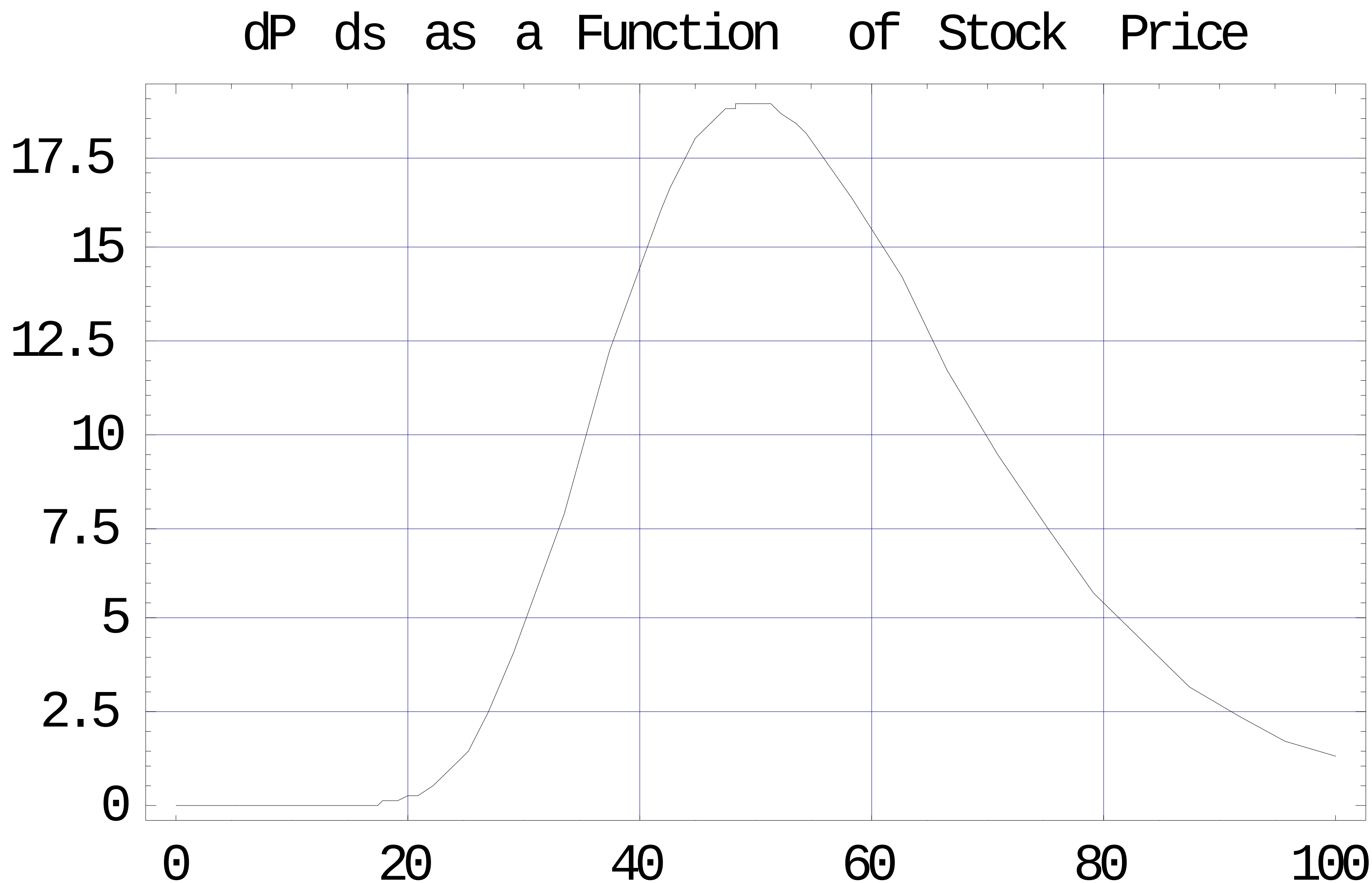
$$\frac{\partial P}{d\sigma} = S[n(-d_1)\frac{\partial d_1}{\partial \sigma} - \frac{e^{-d_1\sigma\sqrt{t}+rt+.5\sigma^2 t - rt - .5\left(\sigma\sqrt{t}-d_1\right)^2}}{\sqrt{2\pi}}(\frac{\partial d_1}{\partial \sigma} - \sqrt{t})]$$

$$= Sn(-d_1)[\frac{\partial d_1}{\partial \sigma} - (\frac{\partial d_1}{\partial \sigma} - \sqrt{t})]$$

$$= Sn(-d_1)\sqrt{t}. \qquad (40)$$

Thus (as indicated in (40) above), $\partial P/\partial \sigma > 0$.

Comparative Statics



Comparative Statics

Derivative	Call Option	Put Option
$\frac{\partial C}{\partial S}$ and $\frac{\partial P}{\partial S}$ $(delta)$	$\frac{\partial C}{\partial S} = N(d_1) > 0$	$\frac{\partial P}{\partial S} = \text{-}N(\text{-}d_1) < 0$
$\frac{\partial C}{\partial K}$ and $\frac{\partial P}{\partial K}$	$\frac{\partial C}{\partial K} = \text{-}e^{-rt}N(d_2) < 0$	$\frac{\partial P}{\partial K} = e^{-rt}N(\text{-}d_2) > 0$
$\frac{\partial C}{\partial r}$ and $\frac{\partial P}{\partial r}$ (rho)	$\frac{\partial C}{\partial r} = tKe^{-rt}N(d_2) > 0$	$\frac{\partial P}{\partial r} = \text{-}tKe^{-rt}N(\text{-}d_2) < 0$
$\frac{\partial C}{\partial t}$ and $\frac{\partial P}{\partial t}$	$\frac{\partial C}{\partial t} = rKe^{-rt}N(d_2) + Sn(d_1)\frac{.5\sigma}{\sqrt{t}} > 0$	$\frac{\partial P}{\partial t} = -rKe^{-rt}N(-d_2) + Sn(d_1)\frac{.5\sigma_k}{\sqrt{t}} \overset{?}{<>} 0$
$theta$	$\text{-}\frac{\partial C}{\partial t} = -rKe^{-rt}N(d_2) - Sn(d_1)\frac{.5\sigma}{\sqrt{t}} < 0$	$\text{-}\frac{\partial P}{\partial t} = rKe^{-rt}N(-d_2) - Sn(d_1)\frac{.5\sigma_k}{\sqrt{t}} \overset{?}{<>} 0$
$\frac{\partial C}{\partial \sigma}$ and $\frac{\partial P}{\partial \sigma}$ $(vega)$	$\frac{\partial C}{\partial \sigma} = Sn(d_1)\sqrt{t} > 0$	$\frac{\partial P}{\partial \sigma} = Sn(-d_1)\sqrt{t} > 0$